

The Semiclassical Limit of Loop Quantum Cosmology

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Abstract

The continuum and semiclassical limits of isotropic, spatially flat loop quantum cosmology are discussed, with an emphasis on the role played by the Barbero–Immirzi parameter γ in controlling space-time discreteness. In this way, standard quantum cosmology is shown to be the simultaneous limit $\gamma \rightarrow 0$, $j \rightarrow \infty$ of loop quantum cosmology. Here, j is a label of the volume eigenvalues, and the simultaneous limit is technically the same as the classical limit $\hbar \rightarrow 0$, $l \rightarrow \infty$ of angular momentum in quantum mechanics. Possible lessons for semiclassical states at the dynamical level in the full theory of quantum geometry are mentioned.

1 Introduction

Quantum geometry [1, 2], one of the candidates for a quantum theory of gravity, faces two main problems: the understanding of its dynamics and of its classical limit. At present there are candidates for a quantization of the Hamiltonian constraint [3, 4], which governs the dynamics, and proposals for the construction of semiclassical states at the kinematical level [5, 6, 7]. Both issues are complicated not only by conceptual problems, but also by the fact that dealing with a general state without any symmetry is technically difficult. In the first case (understanding the dynamics), symmetric models [8] of homogeneous geometries [9, 10] have already proved to be useful [11, 12]. In this note, we will discuss the issue of semiclassical states at the dynamical level.

A common concept of quantum mechanics in this context is the WKB approximation. Intuitively, this gives an approximate solution to the evolution equation of a wave function in a regime where the wave propagation can be described by the motion of a particle (analogous to the ray approximation in optics). Its basic condition is that there is a well-defined, locally almost constant wave length in a neighborhood of any point (which lies in superspace in the case of gravity). Already here, we can see that there are necessary modifications in the case of quantum geometry: since superspace is now discrete [13, 14],

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the wavelength cannot be smaller than a certain scale proportional to the Planck length l_P . At this point the Barbero–Immirzi parameter γ enters because it determines this scale as $\sqrt{\gamma}l_P$, which can, e.g., be seen in the isotropic volume spectrum [15]

$$V_j = (\gamma l_P)^{\frac{3}{2}} \sqrt{\frac{1}{27}j(j + \frac{1}{2})(j + 1)} \quad , \quad j \in \frac{1}{2}\mathbb{N}_0. \quad (1)$$

Similarly, γ enters the spectra of other geometric operators and thus controls the discreteness of geometry. Its physical magnitude can be fixed from calculations of black hole entropy [16, 17] which yields a value of the order one. Note that, contrary to widespread statements in the literature, the existence of a length scale $l_P = \sqrt{\kappa\hbar}$ does not by itself guarantee space-time discreteness, as evidenced by standard quantum cosmology [18, 19] which has continuous space (scale factor) and time. (The same remark applies to quantum electrodynamics in which the existence of a fundamental charge $q = \sqrt{\alpha\hbar}$ does not by itself imply that the charge spectrum must be discrete.) Only a non-zero value of γ in quantum geometry and loop quantum cosmology leads to a discrete volume spectrum and discrete time evolution [20]. In fact, we will show here in the isotropic, spatially flat case that standard quantum cosmology is the $\gamma \rightarrow 0$ (combined with $j \rightarrow \infty$) limit of loop quantum cosmology (just as, e.g., Newtonian mechanics is the $c^{-1} \rightarrow 0$ limit of special relativity). In this way, γ acquires the role of a useful and potentially fundamental parameter: Classical equations of motion can be derived in a combined continuum and semiclassical limit; in addition to ordinary quantum corrections which vanish for $\hbar \rightarrow 0$, there will be new corrections due to the underlying discreteness which vanish for $\gamma \rightarrow 0$ even if \hbar is fixed.

2 Isotropic Canonical Quantum Cosmology

As is usual in quantum cosmology, we will work in a metric (or triad) representation for the wave function ψ of a universe in standard quantum cosmology. More precisely, since the fundamental metrical object in quantum geometry is a densitized triad which in our case can be written as $E_i^a = p\Lambda_i^I X_I^a$ with an internal $SU(2)$ -triad Λ_i^I and left invariant (with respect to the homogeneity group) vector fields X_I^a , a wave function will be represented as a function $\psi(p, \phi)$ where ϕ denotes matter fields. Standard quantum cosmology only makes sense in the regime of positive p (whereas in loop quantum cosmology it is possible to evolve a state to negative p without encountering a singularity [11]), where the relation to the scale factor a and space volume V is given by $p = a^2 = V^{\frac{2}{3}}$.

Because the spectrum of \hat{p} is discrete in quantum geometry, the analog of $\psi(p, \phi)$ in loop quantum cosmology is a discrete wave function $s_n(\phi)$ defined for integers $n \in \mathbb{Z}$. Using the identity $|n| = 2j + 1$ and (1) one obtains the relation

$$n = 6\gamma^{-1}l_P^{-2}p \quad (2)$$

for large positive $n \gg 1$.

As canonically conjugate momentum to p we have the connection coefficient c (from $A_a^i = c\Lambda_I^i \omega_a^I$ with left invariant one-forms ω_a^I dual to X_I^a) fulfilling $\{c, p\} = \frac{1}{3}\gamma\kappa$ ($\kappa = 8\pi G$

is the gravitational constant). For spatially flat cosmological models c is proportional to the extrinsic curvature of a spatial slice. In these variables, the Hamiltonian constraint is given by

$$H = -6\kappa^{-1}\gamma^{-2}c^2\sqrt{p}, \quad (3)$$

which is obtained¹ by adding the negative of the “Euclidean part” $-H^{(E)} = 6\kappa^{-1}c^2\sqrt{p}$ and the “extrinsic curvature part” $P = -6(1 + \gamma^{-2})\kappa^{-1}c^2\sqrt{p}$.

Note that H is proportional to γ^{-2} thanks to the fact that the γ -independent parts in $H^{(E)}$ and P cancel. This leads to a γ -independent Hamiltonian constraint equation in standard quantum cosmology, which is obtained by quantizing $\hat{c} = -\frac{1}{3}i\gamma l_P^2 d/dp$,

$$\kappa \hat{H}_{\text{WdW}} \psi(p, \phi) \equiv \frac{2}{3} l_P^4 \frac{d^2}{dp^2} (\sqrt{p} \psi(p, \phi)) = -\kappa \hat{H}_\phi \psi(p, \phi) \quad (4)$$

using an arbitrary matter Hamiltonian \hat{H}_ϕ (which also depends on p). In general, however, the constraint operator will be γ -dependent when the conditions of spatial flatness or homogeneity are dropped; we will comment on the implications later.

In loop quantum cosmology, which is closer to the full theory of quantum cosmology, the constraint equation looks more complicated. It is directly derived from an adaptation of Thiemann’s operator [3] to isotropy [21] and takes the form of a difference equation for $s_n(\phi)$ (for details we refer to [10]):

$$\begin{aligned} \kappa(\hat{H}s)_n(\phi) &= 3\gamma^{-1}l_P^{-2} \left(\frac{1}{4}(1 + \gamma^{-2}) \left(V_{\frac{1}{2}(n+8)} - V_{\frac{1}{2}(n+8)-1} \right) k_{n+8}^+ k_{n+4}^+ s_{n+8}(\phi) \right. \\ &\quad - \left(V_{\frac{1}{2}(n+4)} - V_{\frac{1}{2}(n+4)-1} \right) s_{n+4}(\phi) \\ &\quad - 2 \left(V_{\frac{1}{2}n} - V_{\frac{1}{2}n-1} \right) \left(\frac{1}{8}(1 + \gamma^{-2})(k_n^- k_{n+4}^+ + k_n^+ k_{n-4}^-) - 1 \right) s_n(\phi) \\ &\quad - \left(V_{\frac{1}{2}(n-4)} - V_{\frac{1}{2}(n-4)-1} \right) s_{n-4}(\phi) \\ &\quad \left. + \frac{1}{4}(1 + \gamma^{-2}) \left(V_{\frac{1}{2}(n+8)} - V_{\frac{1}{2}(n+8)-1} \right) k_{n-8}^- k_{n-4}^- s_{n-8}(\phi) \right) \\ &= -\kappa \hat{H}_\phi(n) s_n(\phi). \end{aligned} \quad (5)$$

Here, the coefficients k_n^\pm , whose explicit form in terms of the volume eigenvalues (1) can be found in [10], come from the extrinsic curvature operator and rapidly approach the value one for large positive n . At large $n \gg 1$, the matter Hamiltonian $\hat{H}_\phi(n)$ is trivially related to that of standard quantum cosmology by (2), and we will focus on the gravitational part \hat{H} in what follows.

¹Recall that in the full theory the Lorentzian Hamiltonian constraint, multiplied by the determinant of the co-triad, can be decomposed into a term which is quadratic in the connection and triad and a non-polynomial term (see formulae (9), (10)).

3 The $\gamma \rightarrow 0$ Limit

The fundamental difference between loop and standard quantum cosmology is the space-time discreteness of the former framework, controlled by the parameter γ (not just by l_P since this scale is present and non-zero in both theories). If we formally take the limit $\gamma \rightarrow 0$, $n \rightarrow \infty$ in such a way that $p(n, \gamma) = \frac{1}{6}\gamma l_P^2 n$ according to (2) is fixed, arbitrary values of p are allowed and the discrete p -spectrum approaches a continuous one. In what follows, $\gamma \rightarrow 0$ will always be understood as this simultaneous limit.

We will now show that in this limit the constraint equation (5) gives (4) if we identify the wave functions $\psi(p, \phi) = s_{n(p)}(\phi)$ at any value of n for a fixed $\gamma \neq 0$ (but only large values $n \gg 1$ become important in the limit). First, we note that the coefficients k_n^\pm approach one for large n and so can be dropped. Next, we introduce a new discrete function

$$t_n(\phi) := \gamma^{-1} l_P^{-2} \left(V_{\frac{1}{2}n} - V_{\frac{1}{2}n-1} \right) s_n(\phi)$$

which in the above limit with the identification of $s_n(\phi)$ and $\psi(p, \phi)$ reduces to $\frac{1}{2}\sqrt{p}\psi(p, \phi) =: \tilde{\psi}(p, \phi)$. The new wave function t is subject to the constraint equation

$$3 \left(\frac{1}{4}(1 + \gamma^{-2})(t_{n+8} - 2t_n + t_{n-8}) - (t_{n+4} - 2t_n + t_{n-4}) \right) = -\gamma l_P^2 \left(V_{\frac{1}{2}n} - V_{\frac{1}{2}n-1} \right)^{-1} \kappa \hat{H}_\phi(n) t_n.$$

Now we use (2) to identify t_{n+k} with $\tilde{\psi}(p + \Delta p(k))$ for any $k \in \mathbb{Z}$, $k \ll n$ and Taylor expand assuming γ to be small:

$$t_{n+k} = \tilde{\psi}\left(p + \frac{1}{6}k\gamma l_P^2\right) = \tilde{\psi}(p) + \frac{1}{6}k\gamma l_P^2 \tilde{\psi}'(p) + \frac{1}{72}k^2\gamma^2 l_P^4 \tilde{\psi}''(p) + O(\gamma^3).$$

The gravitational part of the constraint can be written as

$$\begin{aligned} & \frac{1}{4}(1 + \gamma^{-2})(t_{n+8}(\phi) - 2t_n(\phi) + t_{n-8}(\phi)) - (t_{n+4}(\phi) - 2t_n(\phi) + t_{n-4}(\phi)) \\ &= \frac{1}{4}(1 + \gamma^{-2}) \cdot \frac{16}{9}\gamma^2 l_P^4 \frac{d^2}{dp^2} \tilde{\psi}(p, \phi) - \frac{4}{9}\gamma^2 l_P^4 \frac{d^2}{dp^2} \tilde{\psi}(p, \phi) + O(\gamma^4) \\ &= \frac{4}{9}l_P^4 \frac{d^2}{dp^2} \tilde{\psi}(p, \phi) + O(\gamma^4) \end{aligned}$$

which in the limit $\gamma \rightarrow 0$ yields (4):

$$\frac{4}{3}l_P^4 \frac{d^2}{dp^2} \tilde{\psi}(p, \phi) = \frac{2}{3}l_P^4 \frac{d^2}{dp^2} (\sqrt{p}\psi(p, \phi)) = -\kappa \hat{H}_\phi \psi(p, \phi).$$

We see that standard quantum cosmology (in its full range of all positive values of p) is the $\gamma \rightarrow 0$ limit of loop quantum cosmology, which shows the role of γ as a useful constant controlling the space-time discreteness. Of course, the physical value of γ is non-zero [16, 17] implying that standard quantum cosmology is valid only in certain regimes at large volume where the discreteness does not matter (the above expansion of t_{n+k} can still be done if $p \gg l_P^2$ and ψ is not wildly varying at the Planck scale, irrespective of

the value of γ). In general, there will be γ -corrections to standard quantum cosmology which can be derived from loop quantum cosmology. Only at very small volumes, close to the classical singularity, will it not be sufficient just to include correction terms in an effective Hamiltonian. In this regime, where standard and loop quantum cosmology differ drastically, a discrete formulation is inevitable.

4 The Semiclassical Limit

One way to demonstrate the correct classical limit of a quantum theory consists in introducing the Wigner function [22] (adapted here to our notation for an isotropic cosmological model; see also [23] for an appearance of the Wigner function in standard quantum cosmology)

$$W(p, c) = \int du \overline{\psi(p - \frac{1}{2}l_P^2 u)} \exp(-icu) \psi(p + \frac{1}{2}l_P^2 u)$$

which associates a distribution on the classical phase space to any state ψ . It is a probability distribution (i.e., a non-negative function) only in the classical limit $\hbar \rightarrow 0$, in which case the quantum evolution equation $\hat{H}\psi = 0$ (written as a Hamiltonian constraint for a system with internal time) reduces to the classical Liouville equation $\{H, W\} = 0$ (see, e.g., [24]). For a generalization to a discrete configuration space, on which $\frac{1}{2}u$ may not be defined for all allowed u , the form

$$\tilde{W}(p, c) = \overline{\psi(p)} \int du \exp(-icu) \psi(p + l_P^2 u) \quad (6)$$

will be more suitable. Its properties differ from those of W only at orders of \hbar or higher which are irrelevant for the semiclassical limit.

Wave packets which correspond to a single particle state (rather than a statistical ensemble) in the classical limit can be constructed by using the WKB approximation for a state ψ written in the form $\psi(p) = C(p) \exp(\pm i\hbar^{-1}S(p))$ with $S(p) = S_0(p) + O(\hbar^2)$, where S_0 will turn out to be the classical action, and an \hbar -independent function C . An expansion of $\psi(p \pm \frac{1}{2}l_P^2 u)$ shows that a state ψ in the WKB form has a Wigner function $W(p, c) = |C(p)|^2 \delta(c \mp dS/dp) + O(\hbar)$ which is peaked about classical solutions given by $c(p) = dS/dp$.

In our case of (4) we choose an ansatz $\psi(p, \phi) = p^{-\frac{1}{2}} C(p) \exp(\pm i\hbar^{-1}S_0(p)) \xi(\phi)$ where ξ is a (p -dependent) eigenstate of the matter Hamiltonian: $\hat{H}_\phi(p)\xi = E(p)\xi$. This means that only the gravitational part is treated semiclassically, whereas matter is in a quantum state depending adiabatically on p . Then we can neglect the p -dependence of ξ , and (4) implies

$$\frac{2}{3}(\kappa S'_0)^2 - \kappa p^{-\frac{1}{2}} E(p) \mp \frac{2}{3} i \kappa l_P^2 (S''_0 + 2C^{-1} S'_0 C') + O(l_P^4) = 0,$$

and so $S_0(p)$ fulfills the Hamilton–Jacobi equation for the gravitational background with matter energy $E(p)$, and we have $C(p) = S'_0(p)^{-\frac{1}{2}} = \sqrt{\frac{1}{3}\gamma\kappa c(p)^{-1}}$. Note that unlike a kinematical semiclassical state, a dynamical semiclassical state cannot be peaked about

one phase space point. In particular in an internal time formulation, a peak in the phase space function chosen as internal time cannot be allowed.

The WKB approximation is valid in regimes where $\hbar \log C(p)$ is not strongly varying compared to $S_0(p)$, and thus the $O(l_P^2)$ -term is in fact a small correction to the Hamilton–Jacobi equation. Then,

$$2 \left| \hbar (S'_0)^{-1} \frac{d}{dp} \log C \right| = \left| \frac{\hbar S''_0}{(S'_0)^2} \right| = \left| \frac{d\lambda}{dp} \right| \ll 1 \quad (7)$$

and $\psi \sim \exp(\pm i \hbar^{-1} S_0(p))$ can locally be written as a wave with constant wave length $\lambda = \hbar (S'_0)^{-1}$ defined by

$$\hbar^{-1} S_0(p_0 + \Delta p) = \hbar^{-1} S_0(p_0) + \hbar^{-1} S'_0(p_0) \Delta p + O(\Delta p^2) =: \hbar^{-1} S_0(p_0) + \lambda^{-1} \Delta p + O(\Delta p^2).$$

Since ψ is a wave function on (mini-)superspace and λ gives the oscillation length in p (therefore, λ has dimension length²), which is discrete at a fundamental level, we obtain an additional condition: the oscillation length cannot be smaller than the smallest possible scale, which gives a new condition

$$|\lambda| \gg l_P^2 \quad (8)$$

in addition to (7). Otherwise, a state would have strong variation between successive values of n violating the pre-classicality condition of [12]. (Note that this condition may even be violated at large volume, e.g. in the presence of a cosmological constant Λ in which case we have an action $S_0 \propto \sqrt{\Lambda} V = \sqrt{\Lambda p} \cdot p$ leading to a wave length which tends to zero for large p . This kind of infrared problem, however, is an artefact of the minisuperspace approximation — curvature enters the formalism in the space integrated form which can be large even if the local curvature scale is small — and can be ignored here.)

When the two conditions (7), (8) for a semiclassical behavior are fulfilled, the correct semiclassical limit of loop quantum cosmology follows from the discussion above: the continuum limit $\gamma \rightarrow 0$ leads to standard quantum cosmology whose semiclassical limit $\hbar \rightarrow 0$ is demonstrated using the Wigner function. It is also possible to sidestep the Wheeler–DeWitt formulation by defining a Wigner function directly for loop quantum cosmology. A straightforward generalization of (6) is

$$W_n(c) = (\sqrt{2} \sin(\tfrac{1}{2}c))^{-1} s_n \sum_{k \in \mathbb{Z}} T_k(c) s_{n+k}$$

which is a distribution function, associated with a state s_n , on the (partially discrete) space with coordinates (n, c) . Because of the discreteness, the integral in (6) is replaced by a sum. Moreover, instead of the Fourier transform we use the functions $T_n(c) = (\sqrt{2} \sin(\tfrac{1}{2}c))^{-1} \exp(\tfrac{1}{2}inc)$ which are eigenstates of \hat{p} and are used in a transformation between the triad and the connection representation [10]. The factor $(\sqrt{2} \sin(\tfrac{1}{2}c))^{-1}$ in $W_n(c)$ accounts for the non-trivial measure in the c -representation. The classical limit again is derived by an expansion of the constraint equation for $W_n(c)$ in both γ and \hbar . Note that γ always appears multiplied by \hbar (or l_P^2), and so the discreteness corrections also disappear

if we perform the limit $\hbar \rightarrow 0$ by itself, fixing γ . However, if we are interested not only in the classical limit but also in corrections, two different types of terms occur: one caused by the discreteness (depending on γ) and one purely from quantum theory (γ -independent). Unlike before, the continuum limit $\gamma \rightarrow 0$ is now done at the phase space level and standard quantum comology does not appear as an intermediate step.

We can directly apply the WKB prescription to the discrete wave function $s_n \sim C(n) \exp(\pm i\hbar^{-1}S(p(n)))$. Because we already know that the constraint equations of standard and loop quantum cosmology agree up to γ -corrections, this state is an approximate solution to the constraint (5) up to \hbar - and γ -corrections. Locally (in a neighborhood of a given value n_0), for any solution s_n of (5) there is a solution $\psi(p)$ of (4) which approximates s_n . But the γ -corrections in general add up when solving the difference equation, and so two solutions s_n and $\psi(p)$ can differ, e.g. by a phase shift, away from n_0 . Nevertheless, the leading order (in γ and \hbar) of the Wigner functions associated with both states is the same, which is a direct consequence of the expansion being sensitive only to the local behavior of a wave function.

5 Conclusions

We have shown that one can perform the semiclassical limit of isotropic, spatially flat loop quantum cosmology in a two-step procedure leading to the correct classical behavior: the first step is the continuum limit which can be formulated as $\gamma \rightarrow 0$, followed by a second step which is the usual semiclassical limit of quantum mechanics. At the intermediate level, one obtains standard quantum cosmology as the continuum limit of loop quantum cosmology. This also shows how to define semiclassical (WKB) states at the dynamical level of isotropic quantum cosmology whose correlations are peaked about the classical ones.

We conclude with a few remarks about lessons for a possible generalization to the full theory of quantum geometry. In this case the inhomogeneity is manifested in the appearance of arbitrary graphs to which states are associated. The continuum limit can then no longer be performed by only $\gamma \rightarrow 0$ together with labels going to infinity, but has to be extended by a prescription of how to shrink the graphs to continuous objects (analogous to a vanishing lattice spacing in lattice gauge theories). In this process the number of vertices will become infinite, and the standard methods of quantum geometry become ill-defined; this may be related to the fact that the Wheeler–DeWitt quantization of gravity is only defined formally for inhomogeneous geometries. In this respect, the second strategy of the previous section, defining a Wigner function directly on the discrete phase space, will be more suitable since it does not make use of an intermediate Wheeler–DeWitt quantization.

In addition to the issue of graphs, the full Hamiltonian constraint operator is more complicated than that of isotropic models. This is in part due to the appearance of arbitrary graphs, but most importantly due to a complicated volume spectrum which is not known explicitly. A possible route consists in again using reduced models, this time

mid-superspace models which are inhomogeneous (e.g., spherically symmetric models or cylindrical waves which both have a single inhomogeneous axis). This allows to investigate the new aspects of inhomogeneous states without dealing directly with the full constraint.

Some observations can already be inferred from the classical form of the Hamiltonian constraint with Euclidean part

$$\kappa \det(e_I^i) H^{(E)} = -\epsilon_{ijk} F_{ab}^i E_j^a E_k^b = -2 \left(\epsilon_{ijk} \partial_{[a} A_{b]}^i + A_{[a}^j A_{b]}^k \right) E_j^a E_k^b \quad (9)$$

and extrinsic curvature part

$$\kappa \det(e_I^i) P = -2(1 + \gamma^{-2})(A_a^i - \Gamma_a^i)(A_b^j - \Gamma_b^j) E_i^{[a} E_j^{b]} \quad (10)$$

which yield the constraint

$$\begin{aligned} H &= -H^{(E)} + P \\ &= 2\kappa^{-1} \det(e_I^i)^{-1} \left(\epsilon_{ijk} \partial_a A_b^k - \gamma^{-2} A_{[a}^i A_{b]}^j - (1 + \gamma^{-2}) \left(\Gamma_{[a}^i \Gamma_{b]}^j - 2A_{[a}^i \Gamma_{b]}^j \right) \right) E_i^a E_j^b. \end{aligned}$$

In the isotropic, spatially flat constraint only the middle term is present which leads to a γ -independent constraint after quantizing in a triad representation since A_a^i becomes an operator proportional to γ . The first term, on the other hand, would be γ -dependent, but also is sensitive to the continuum limit of graphs (due to the partial derivative). This suggests to link the graph continuum limit to the $\gamma \rightarrow 0$ limit. The last term containing Γ_a^i looks problematic since it diverges in the $\gamma \rightarrow 0$ limit. However, locally one can always choose coordinates such that $\Gamma_a^i = 0$ eliminating this term. Since invariance under coordinate transformations is incorporated by the diffeomorphism constraint, this term may play a role in understanding the relation and algebra of the quantized diffeomorphism and Hamiltonian constraints.

As another application one can use similar ideas to find a relation between the loop and Fock space quantizations of Maxwell theory, which also allows an analog of the Barbero–Immirzi parameter (usually set to one by using the known value of the fundamental charge [25]). This would lead to a transformation from a quantization with discrete charges to one with a continuous charge spectrum, different from [26].

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